Model of a two–transverse mode laser with injected signal

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Abstract

We derive a simple model for a two transverse mode laser (that considers the TEM_{00} and TEM_{10} modes) in which an injected signal with the shape of the TEM_{10} mode but a frequency close to that of the TEM_{00} mode is injected.

Here we derive a model for a two–transverse mode laser with injected signal. More specifically, we do consider an incoherently pumped homogeneously broadened laser, in a ring cavity configuration in which a particular signal is injected from the outside. We assume that the laser works reasonably close to resonance with the TEM_{00} mode, but the injected field has not the shape of the TEM_{00} mode but the shape of the TEM_{10} mode. The frequency of the injected field is however assumed to be close to that of the TEM_{00} mode. We first develop a two–mode model which is valid for any value of the decay rates. Later on, we particularize this model to class–A lasers, i.e., lasers in which the material variables decay much faster than the intracavity field. We end up with a single complex equation for the TEM_{00} mode which is of the Stuart–Landau type.

1 General model

The starting point are the laser equations for a large Fresnel number ring–cavity with spherical mirrors filled with a collection of homogeneously broadened two–level atoms that are incoherently pumped. These equations can be found, e.g., in [1, 2]. To this model equations we add an injected signal with arbitrary spatial

structure. In the single longitudinal mode and uniform field approximations these equations can be written in dimensionless form as

$$\frac{\partial}{\partial t}F\left(\mathbf{r},t\right)=-\left(1+i\theta\right)F+ia\mathcal{L}F+2CP+F_{\mathrm{in}}\left(\mathbf{r}\right), \tag{1a}$$

$$\frac{\gamma_{\mathrm{c}}}{\gamma_{\perp}}\frac{\partial}{\partial t}P\left(\mathbf{r},t\right)=-\left(1+i\Delta\right)P+FD, \tag{1b}$$

$$\frac{\gamma_{\mathrm{c}}}{\gamma_{||}}\frac{\partial}{\partial t}D\left(\mathbf{r},t\right)=-D+\chi-\mathrm{Re}\left(F^{*}P\right). \tag{1c}$$

In these equations F and P are proportional to the slowly varying complex amplitudes of the laser electric field and the medium polarization, and D is proportional to the atomic inversion, which decay at rates γ_{c}, $\gamma_{\perp}$, and $\gamma_{||}$, respectively. Time t is measured in units of γ_{c}^{-1}. The external monochromatic injection is defined by its complex amplitude, proportional to $F_{\mathrm{in}}\left(\mathbf{r}\right)$, and by its frequency ω_{in}, which serves as the frequency frame in which the equations have been written, so the detunings read

$$\theta=\frac{\omega_{\mathrm{c}}-\omega_{\mathrm{in}}}{\gamma_{\mathrm{c}}},\ \Delta=\frac{\omega_{\mathrm{a}}-\omega_{\mathrm{in}}}{\gamma_{\perp}}, \tag{2}$$

where ω_{c} is the cavity frequency of the fundamental, TEM_{00}, transverse mode, and ω_{a} is the atomic frequency. C is the usual cooperativity parameter and χ sets the value of the inversion in the absence of fields.

Finally operator $\mathcal{L}=\left(\frac{1}{4}\nabla^{2}-r^{2}+1\right)$ accounts for the modal structure of the resonator, with $\nabla^{2}=\left(\partial^{2}/\partial x^{2}+\partial^{2}/\partial y^{2}\right)$ the Laplacian acting on the transverse coordinates $\mathbf{r}=(x,y)$, which are measured in units of the TEM_{00} beam waist radius, and a is the transverse mode spacing (measured in units of γ_{c}). We note that the set of Hermite-Gauss (TEM_{mn}) modes

$$\Psi_{m,n}\left(\mathbf{r}\right)=\frac{1}{\sqrt{2^{m+n-1}m!n!\pi}}H_{m}\left(\sqrt{2}x\right)H_{n}\left(\sqrt{2}y\right)e^{-r^{2}}, \tag{3}$$

are eigenfunctions of $\mathcal{L}$,

$$\mathcal{L}\Psi_{m,n}\left(\mathbf{r}\right)=-\left(m+n\right)\Psi_{m,n}\left(\mathbf{r}\right). \tag{4}$$

The Hermite-Gauss modes verify the orthonormality condition

$$\int\mathrm{d}^{2}\mathbf{r}\ \Psi_{m,n}\left(\mathbf{r}\right)\Psi_{m',n'}\left(\mathbf{r}\right)=\delta_{m,m'}\delta_{n,n'}, \tag{5}$$

and form a basis for functions defined on the transverse plane, hence any function $G\left(\mathbf{r}\right)$ can be expanded into it as

$$G\left(\mathbf{r},t\right)=\sum_{m,n}g_{m,n}\left(t\right)\Psi_{m,n}\left(\mathbf{r}\right), \tag{6}$$

where $g_{m,n}\left(t\right)$ are the modal coefficients.

The above is the starting model from which we pass to derive a two–mode model.

2 Two mode class C laser model

We consider the situation in which the (normalized) frequency spacing between the transverse modes, a, which only depends on the cavity geometry, is large ($a \gg 1$) while the (normalized) detunings θ, and Δ are, at most, of order 1. Without injection, it is evident that under the above conditions only the fundamental, Gaussian TEM_{00} mode will be excited above threshold. Nevertheless the injected field is chosen to have the shape of the TEM_{10} mode,

$$F_{\mathrm{in}}(\mathbf{r}) = f_{\mathrm{in}} \Psi_{1,0}(\mathbf{r}), \tag{7}$$

with f_{in} the injected field amplitude, which we take as a constant parameter. Hence, apart from the TEM_{00} mode, the TEM_{10} mode will be oscillating as well, and then it is feasible to approximate the total field as

$$F(\mathbf{r}, t) = f_0(t)\,\Psi_{0,0}(\mathbf{r}) + f_1(t)\,\Psi_{1,0}(\mathbf{r}), \tag{8}$$

at least when the system is not very far from threshold. Of course this approximation will be invalid if the injected field has a frequency which is close to that of the TEM_{10} mode, and in this case more mode families should be taken into account. We shall remind later on the smallness of the injected signal detuning as compared with the frequency spacing between the transverse modes. In a similar way we approximate

$$P(\mathbf{r}, t) = p_0(t)\,\Psi_{0,0}(\mathbf{r}) + p_1(t)\,\Psi_{1,0}(\mathbf{r}), \tag{9a}$$
$$D(\mathbf{r}, t) = d_0(t)\,\Psi_{0,0}(\mathbf{r}) + d_1(t)\,\Psi_{1,0}(\mathbf{r}). \tag{9b}$$

By substituting the above expansions into the original model equations, and projecting onto the different modes, the following evolution equations for the amplitudes are found

$$\frac{\mathrm{d}}{\mathrm{d}t} f_0(t) = -f_0 - i\theta f_0 + 2Cp_0, \tag{10a}$$
$$\frac{\mathrm{d}}{\mathrm{d}t} f_1(t) = -f_1 - i(\theta + a) f_1 + 2Cp_1 + f_{\mathrm{in}}, \tag{10b}$$
$$\frac{\gamma_{\mathrm{c}}}{\gamma_{\perp}} \frac{\mathrm{d}}{\mathrm{d}t} p_0(t) = -p_0 - i\Delta p_0 + \eta f_0 d_0 + \frac{2}{3}\eta f_1 d_1, \tag{10c}$$
$$\frac{\gamma_{\mathrm{c}}}{\gamma_{\perp}} \frac{\mathrm{d}}{\mathrm{d}t} p_1(t) = -p_1 - i\Delta p_1 + \frac{2}{3}\eta (f_0 d_1 + f_1 d_0), \tag{10d}$$
$$\frac{\gamma_{\mathrm{c}}}{\gamma_{||}} \frac{\mathrm{d}}{\mathrm{d}t} d_0(t) = -d_0 + \chi_0 - \eta\,\mathrm{Re}(f_0^* p_0) - \frac{2}{3}\eta\,\mathrm{Re}(f_1^* p_1), \tag{10e}$$
$$\frac{\gamma_{\mathrm{c}}}{\gamma_{||}} \frac{\mathrm{d}}{\mathrm{d}t} d_1(t) = -d_1 - \frac{2}{3}\eta\,\mathrm{Re}(f_0^* p_1 + f_1^* p_0). \tag{10f}$$

where $\eta = \frac{2\sqrt{2}}{3\sqrt{\pi}}$.

These are our two-transverse mode class–C model equations. It is convenient to recall that the range of validity of these equations is limited to large values of the transverse mode spacing ($a \gg 1$) and to small values of the detunings θ and Δ (at most of order a^0), and not large pump nor large injected signal.

3 Two-mode Class A laser model

In order to simplify the problem as much as possible we consider a class A laser, defined by

$$\frac{\gamma_{\mathrm{c}}}{\gamma_{\perp}}, \frac{\gamma_{\mathrm{c}}}{\gamma_{||}} \ll 1, \tag{11}$$

and consequently we eliminate adiabatically the medium variables. Furthermore the condition $a \gg 1$ allows eliminating the modal amplitude f_1 as well. Finally, we will consider that the laser is operated close to threshold, as commented, and use a cubic approximation in the fields as usual. With all these elements one can express the model variables p_0, p_1, d_0, d_1, and f_1 in terms of the fundamental mode amplitude f_0 and of the model parameters.

Substitution of these expressions into the equation for f_0 yields finally

$$\frac{\mathrm{d}}{\mathrm{d}t} A\left(t\right) = \alpha A + \beta A^{*} - \left(1 - i\Delta\right) \left|A\right|^{2} A, \tag{12}$$

where the relation between the field A and the Gaussian mode amplitude f_0 is

$$A\left(t\right) = \eta f_0\left(t\right), \tag{13}$$

and the complex parameters α and β in Eq. (12) are given by

$$\begin{aligned}
\operatorname{Re}\left(\alpha\right) &= \frac{r}{1+\Delta^{2}} - 1 - \frac{\left(11-\Delta^{2}\right) a^{2} E^{2}}{5\left(1+\Delta^{2}\right)\left(a+\theta\right)^{2}}, &(14)\\
\operatorname{Im}\left(\alpha\right) &= -\frac{r\Delta}{1+\Delta^{2}} - \theta + \frac{12\Delta a^{2} E^{2}}{5\left(1+\Delta^{2}\right)\left(a+\theta\right)^{2}}, &(15)\\
\beta &= \frac{a^{2} E^{2}}{5\left(1+\Delta^{2}\right)\left(a+\theta\right)^{2}} \left(5+\Delta^{2}-4i\Delta\right), &(16)
\end{aligned}$$

where

$$\begin{aligned}
E &= \frac{2}{3}\sqrt{\frac{6}{5}}\frac{\eta}{a} f_{\mathrm{in}}, &(17\mathrm{a})\\
r &= 2C\eta\chi_0. &(17\mathrm{b})
\end{aligned}$$

Equation (12) constitutes the final model. We note that $A\left(t\right)$ is proportional to the amplitude $f_0\left(t\right)$ of the TEM_{10} mode, and that E is a free parameter proportional to the amplitude of the external injection, f_{in}. The remaining parameters are the detunings θ and Δ, and the pump parameter r, dependent on the usual pump parameter $2C$: It is easy to derive that the threshold value of the free–running laser (with injection $E = 0$) is given by $r = 1 + \Delta^2$, i.e. by

$$2C = \frac{1+\Delta^{2}}{\eta\chi_0}. \tag{18}$$

Let us remind that in the derivation we have assumed that the fields are weak, thus implying that r is close its threshold value, that the decay rates of

the material variables are much larger than that of the intracavity field, and that the intermode spacing is large compared to the cavity decay rate ($a >> 1$). We have also assumed that $2\theta < a$ as for larger cavity detunings it does not make sense the assumption that the laser is working, basically, in the TEM_{00} mode.

We find it worth writing the above Stuart–Landau equation in the special case $\Delta = 0$ (i.e., $\omega_{\mathrm{in}} = \omega_{\mathrm{a}}$), as this can be easily done in the laboratory. In this case the equation simplifies to

$$\frac{\mathrm{d}}{\mathrm{d}t} A\left(t\right) = \left(r - 1 - \frac{11}{5} \frac{a^2 E^2}{\left(a+\theta\right)^2} - i\theta\right) A + \frac{a^2 E^2}{\left(a+\theta\right)^2} A^* - \left|A\right|^2 A. \tag{19}$$

Notice that the infinite resonances appearing in parameters α and β for $\theta = -a$ is an artifact, as in the derivation we have implicitely assumed that the cavity detuning is smaller than half the intermode spacing, i.e., $2\theta < a$.

References

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